A stochastic geometry approach to wideband ad hoc networks with channel variations

Steven Weber
Drexel University, Dept. of ECE
Philadelphia PA 19104
sweber@ece.drexel.edu

Jeffrey G. Andrews
The University of Texas at Austin, Dept. of ECE
Austin TX 78712
jandrews@ece.utexas.edu

***Abstract*— We present a methodology for determining the outage probability of wideband ad hoc networks with random wireless channels. Assuming that the nodes are Poisson distributed and subject to a required SINR constraint, we develop a simple framework that gives upper and lower bounds on the outage probability. These bounds are important in that they can be manipulated to obtain bounds on the transmission capacity, i.e., the maximum permissible spatial density of transmissions ensuring an acceptably low outage probability. In this paper, we derive the outage probability of wireless ad hoc networks under path loss and shadowing, which are the dominant large-scale effects in wideband ad hoc networks. The analytical framework is rooted in stochastic geometry, employing marked point processes, void probabilities, Palm measure, and Campbell's Theorem.**

I. Introduction

The long-term viability of decentralized wireless networking, compared to the more traditional centralized wireless networking, depends largely on the fundamental capabilities of a network of randomly distributed, mutually interfering, wireless nodes. This paper introduces a general framework, extending the framework in [1], for analyzing the effect of fading channels on ad hoc network outage probability as well as the transmission capacity. This framework can be used to compute both upper and lower bounds on the outage probability as well as the allowable intensity of transmitters in the network subject to a target outage probability (QoS constraint).

A. Ad Hoc Network Capacity

Ad hoc network capacity has been a highly active research area particularly since the seminal result of Gupta and Kumar [2], that found that the *transport capacity* of a large random wireless ad hoc network with n nodes scaled as $(n \log n)^{-\frac{1}{2}}$. Numerous interesting extensions of this result have since been derived, including for example [3], [4], and [5] which incorporated respectively mobility, bandwidth, and energy considerations. In these cases the scaling in n can be improved under certain conditions. More recently, the Gupta-Kumar result has been verified from an information theoretic perspective [6] using fairly simple probabilistic methods [7].

A stochastic geometric approach pioneered by Baccelli and others [8], [9] based on Poisson point processes and their accompanying rich body of theory has proven very useful in characterizing many key features of wireless networks. More recent work [10], [11] has used stochastic geometry to study connectivity and throughput scaling bounds for ad hoc networks. The *Transmission Capacity* is a capacity metric that gives the maximum spatial density of achievable transmissions in a wireless ad hoc network, subject to a quality of service constraint [1]. It has been recently used to show the impact of spread spectrum [1], interference cancellation [12], and scheduling [13]. *All our prior work has assumed channels subject to only path loss effects. In this paper, we extend the framework to include shadowing effects.* Path loss and shadowing are the the dominant large-scale effects in wideband ad hoc networks. Our approach employs *marked Poisson point processes*, where the marks are random variables for each transmitter capturing shadowing effects for the channel connecting the transmitter to a reference receiver at the origin.

B. Random Channels in Wireless Ad Hoc Networks

A fundamental difference between ad hoc and centralized networks is the extent to which the node locations affect not only the quality of their own transmission, but that of all the other nodes. For this reason, most prior work on wireless ad hoc network capacity has focused on the geometric aspects of the network, and considered path loss channel models. Formally, it has typically been assumed that

$$P_{ij} = P_j d_{ij}^{-\alpha},$$

where $P_i j$ is the received power at node i from node j, $d_{ij} \geq 1$ is the transmission range between nodes i and j, $\alpha \geq 2$ is the path loss exponent, and P_j is the far-field transmit power at node j (i.e. $P_{ij} = P_j$ at $d_{ij} = 1$). While this simple model successfully captures the spatial interactions between the nodes, it has some important shortcomings. First, wireless channels are known to have random distributions that can change the received power by several orders of magnitude. Hence, a preferable channel model is

$$P_{ij} = P_j d_{ij}^{-\alpha} h_{ij},$$

where h_{ij} is a random variable. This random channel gain h_{ij} can have various statistical distributions in different propagation environments, including lognormal, exponential, Ricean, Nakagami, and compounds of these.

A second motivation for including fading is that many capacity increasing techniques, such as transmit/receive diversity, spatial multiplexing, adaptive modulation, and multiuser diversity depend on variations in the channel (see [14], [15] and the references therein). In order to realistically quantify the gain that these techniques offer in a decentralized wireless network, it is crucial to have a proper capacity framework based on statistical fading channels. Although this paper does not attempt to determine the gain from these diversity techniques, we develop a framework that should prove useful for doing so.

C. A stochastic geometric approach

Our analytical framework for obtaining bounds on the outage probability is rooted in stochastic geometry. In particular, the locations of the transmitting nodes as well as their channel conditions are captured as a marked Poisson point process. We employ Slivnyak's theorem to condition on the presence of a "typical" reference receiver at the origin, permitting us to obtain node-average performance quantities through the associated Palm measure. Campbell's theorem is used to obtain expressions for the expected value and variance of certain aggregate normalized interference terms. Void probabilities are used to compute the lower bound on the outage probability.

The crux of our proof of the bounds lies in our observation that there are two kinds of interfering nodes (as observed by the reference node at the origin): dominant interfering nodes that by themselves can cause outage for the reference receiver, and non-dominant nodes whose interference contribution is individually insufficient for outage. We split the set of interferers into these two groups and are able to exploit the properties of these two groups to obtain good bounds.

The rest of this paper is structured as follows. In Section II we present the mathematical framework for an ad hoc network with channel variations, culminating in Theorem 1 which gives expressions for the outage probability bounds. In Section III we validate our analysis through simulation and numerical results; the plots demonstrate the bounds are reasonably tight. Finally, we conclude in Section IV and propose extensions to this work. Proofs are placed in the Appendix.

II. Transmission Capacity for Random Channels

A. The model

We consider an ad hoc network where at some instant in time the set of transmitters form a homogeneous Poisson point process on the plane of intensity λ. Let $\{X_i \in \mathbb{R}^2, i \in \mathbb{Z}_+\}$ be the locations of the transmitters attempting communication at time t. Our aim is to evaluate the outage probability $p_o(\lambda)$ as a function of the spatial intensity of transmissions λ.

To evaluate the outage probability we will condition on a typical receiver at the origin yielding the Palm distribution for transmitters on the plane [16]. It follows by Slivnyak's Theorem [16] that this conditional distribution also corresponds to a homogenous Poisson point process with the same intensity. We will denote probability and expectation with respect to this distribution by $\mathbb{P}^0$ and $\mathbb{E}^0$ respectively. Because the reference receiver at the origin is "typical" the outage probability and transmission capacity for that receiver are in fact the node-average quantities as well.

Let $r_{\rm tx}$ be a fixed parameter capturing the distance separating each receiver from its associated transmitter. For simplicity we assume all nodes employ a common transmission power denoted $P_{\rm tx}$. An attempted transmission is successful if the received signal to interference ratio (SIR) at the receiver is above a threshold, β. If the received SIR is less than β then the attempted transmission fails, i.e., an outage occurs. Our model does not include an additive noise term. The justification of this decision is primarily to preserve analytical tractability and simplicity, however, in an interference limited network the noise contribution will be minimal [17].

The received power from a transmitter (either signal or interferer) separated by a distance d under a path loss plus shadowing channel model is

$$P_{\rm rx} = P_{\rm tx} K \left(\frac{d_{\rm ref}}{d}\right)^{\alpha} \Psi', \tag{1}$$

where K captures antenna characteristics, $d_{\rm ref}$ is the reference distance, and Ψ' is a lognormal random variable with probability density function

$$f_\Psi(\psi) = \frac{10/\ln 10}{\sqrt{2\pi}\sigma\psi} \exp\left\{-\frac{(10\log_{10}\psi)^2}{2\sigma^2}\right\}, \ \psi \in \mathbb{R}_+. \tag{2}$$

See, e.g., [14]. It is straightforward to establish that:

$$\mathbb{E}[\Psi] = e^{(\frac{\ln 10}{10})^2 \frac{1}{2}\sigma^2}, \mathbb{E}[\Psi^2] = e^{(\frac{\ln 10}{10})^2 2\sigma^2}. \tag{3}$$

The above channel model suffers the problem that the received power may exceed the transmitted power for distances $d < d_{\rm ref}$, which is not physically meaningful. To rectify this we operate under the assumption that the receiver is capable of perfect interference cancellation for any interfering node at distance $d < d_{\rm ref} = 1$ meter.

Let $\Phi(\lambda)' = \{(X_i, \Psi_i'), i \in \mathbb{Z}_+\}$ be a homogeneous marked Poisson point process (MPPP) of spatial intensity λ where the locations $\{X_i\}$ indicate the positions of the transmitters at some time t and the marks $\{\Psi_i'\}$ are the shadowing term for the channel connecting transmitter i to the reference receiver at the origin. By assumption each mark Ψ_i' is independent of the other marks and points, and in particular, is independent of the associated location X_i. Under these assumptions the outage probability for the (typical) reference receiver located at the origin is

$$\begin{aligned} p_o(\lambda) &= \mathbb{P}^0\left(\frac{P_{\rm tx} r_{\rm tx}^{-\alpha}\Psi_0'}{\sum_{i\in\Phi(\lambda)'} P_{\rm tx}|X_i|^{-\alpha}\Psi_i'} \leq \beta\right) \\ &= \mathbb{P}^0\left(\sum_{i\in\Phi(\lambda)} \Psi_i R_i^{-\alpha} \geq \kappa\right). \end{aligned}$$

Here $\Phi(\lambda) = \{(X_i, \Psi_i), i \in \mathbb{N}\}$ is the MPPP obtained by forming the mark ratios $\Psi_i = \Psi_i'/\Psi_0'$ for each transmitter $i = 1, 2, 3, \ldots$, where recall Ψ_0' is the shadowing term for the channel between the reference receiver at the origin and its associated transmitter. Define the constant $\kappa = \frac{1}{\beta r_{\rm tx}^{\alpha}}$. Moreover, $R_i = |X_i|$ is the distance from transmitter i to the origin. The following fact is straightforward to establish:

Fact 1: If Ψ_i' and Ψ_0' are independent lognormal random variables with parameter σ then the ratio $\Psi_i = \Psi_i'/\Psi_0'$ is also a lognormal random variable with parameter $\sqrt{2}\sigma$.

Note that the outage probability is the tail probability of a functional of a marked point process where the marks are the lognormal shadowing ratios of each transmitter over the reference transmitter. Our goal in the next section is to establish bounds on $p_o(\lambda)$.

B. Bounding the outage probability

Our approach to obtaining upper and lower bounds on the outage probability $p_o(\lambda)$ is to split the MPPP $\Phi(\lambda)$ into two separate MPPP's: the first MPPP contains all points whose individual interference contributions are by themselves sufficient to cause an outage at the reference receiver, and the second MPPP contains all the remaining points. We can think of two distinct "modes" in which outages may occur: 1) individual interfering transmitters generate large amounts of interference thereby causing outage, or 2) individual interfering transmitters each generate small amounts of interference which cumulatively is sufficient to cause an outage. The upper bound on the outage probability is obtained by upper bounding a random variable through either a Markov or Chebychev inequality. We present upper bounds for both these inequalities. Roughly speaking, the Chebychev bound is usually tighter than the Markov bound, but also is usually a bit more complex of an expression. As seen in the proof of the Theorem the upper bound is a shot noise process and as such it should also be possible to apply bounds on such processes to our framework, e.g., [18], [19]. This approach is the subject of current work but is not pursued here. The lower, Markov upper, and Chebychev upper bounds will be denoted by $p_l(\lambda), p_u^m(\lambda), p_u^c(\lambda)$ respectively.

Theorem 1: The lower and upper bounds on the outage probability $p_o(\lambda)$ when the channel effects include path loss plus shadowing are given by:

$$\begin{aligned} p_l(\lambda) &= 1-\exp\{-p_\Psi\lambda\}, \\ p_u^m(\lambda) &= 1-\exp\{-p_\Psi\lambda\}+\frac{\mu_\Psi}{\kappa}\lambda, \\ p_u^c(\lambda) &= 1-\exp\{-p_\Psi\lambda\}+\frac{\sigma_\Psi^2\lambda}{(\kappa-\mu_\Psi\lambda)^2}. \end{aligned}$$

The constants $p_\Psi, \mu_\Psi, \sigma_\Psi^2$ are given by:

$$\begin{aligned} p_\Psi &= 2\pi\int_1^\infty \mathbb{P}(\Psi>\kappa r^\alpha) r dr, \\ \mu_\Psi &= 2\pi\int_1^\infty r^{-\alpha}\Big[\int_0^{\kappa r^\alpha}\psi f_\Psi(\psi)d\psi\Big] r dr, \\ \sigma_\Psi^2 &= 2\pi\int_1^\infty r^{-2\alpha}\Big[\int_0^{\kappa r^\alpha}\psi^2 f_\Psi(\psi)d\psi\Big] r dr. \end{aligned}$$

The three constants $p_\Psi, \mu_\Psi, \sigma_\Psi^2$ are difficult to compute numerically in their given form due to the double integrals. The following corollary reduces their computational complexity somewhat.

Corollary 1: When Ψ is a lognormal random variable with parameter ν^2 the constants $p_\Psi, \mu_\Psi, \sigma_\Psi^2$ are given by:

$$\begin{aligned} p_\Psi &= 2\pi\int_1^\infty \bar{F}_Z\Big(\frac{10}{\nu}\log_{10}(\kappa r^\alpha)\Big) r dr, \\ \mu_\Psi &= 2\pi\mathbb{E}[\Psi]\int_1^\infty r^{1-\alpha} F_Z\Big(\frac{\ln(\kappa r^\alpha)-(\frac{\ln 10}{10})^2\nu^2}{(\frac{\ln 10}{10})\nu}\Big) dr, \\ \sigma_\Psi^2 &= 2\pi\mathbb{E}[\Psi^2]\int_1^\infty r^{1-2\alpha} F_Z\Big(\frac{\ln(\kappa r^\alpha)-2(\frac{\ln 10}{10})^2\nu^2}{(\frac{\ln 10}{10})\nu}\Big) dr, \end{aligned}$$

where $F_Z(z)$ is the cumulative distribution function (CDF) for a standard $\mathcal{N}(0,1)$ random variable and $\bar{F}_Z(z) = 1 - F_Z(z)$ is the complementary CDF. The quantities $\mathbb{E}[\Psi]$ and $\mathbb{E}[\Psi^2]$ are given by (3).

Our previous work [1] analyzed bounds on the outage probability for channels with only path loss effects, i.e., the received power from a transmitter (either signal or interferer) separated by a distance d is $P_{\rm rx} = P_{\rm tx} K \left(\frac{d_{\rm ref}}{d}\right)^\alpha$. The resulting bounds on the outage probability are given by the same expressions in Theorem 1 but with different constants p_L, μ_L, σ_L^2 given by:

$$\begin{aligned} p_L &= \pi(\kappa^{-\frac{2}{\alpha}}-1) \\ \mu_L &= \frac{2\pi}{\alpha-2}\kappa^{1-\frac{2}{\alpha}} \\ \sigma_L^2 &= \frac{\pi}{\alpha-1}\kappa^{2(1-\frac{1}{\alpha})} \end{aligned}$$

Substituting these constants and simplifying yields the following bounds:

Corollary 2: The lower and upper bounds on the outage probability for a pure path loss channel are:

$$\begin{aligned} p_l(\lambda) &= 1-\exp\{-\pi(\kappa^{-\frac{2}{\alpha}}-1)\lambda\}, \\ p_u^m(\lambda) &= 1-\exp\{-\pi(\kappa^{-\frac{2}{\alpha}}-1)\lambda\}+\frac{2\pi}{\alpha-2}\kappa^{-\frac{2}{\alpha}}\lambda, \\ p_u^c(\lambda) &= 1-\exp\{-\pi(\kappa^{-\frac{2}{\alpha}}-1)\lambda\}+\frac{\frac{\pi}{\alpha-1}\kappa^{-\frac{2}{\alpha}}\lambda}{(1-\frac{2\pi}{\alpha-2}\kappa^{-\frac{2}{\alpha}}\lambda)^2}. \end{aligned}$$

C. Bounding the transmission capacity

We define the *transmission capacity* as the maximum spatial density of transmissions, $\lambda(\epsilon)$, such that the resulting outage probability is at most ϵ, where $\epsilon \in (0,1)$ is a QoS parameter for the network [1]. Trivially, $p_o(\lambda(\epsilon)) = \epsilon$. The transmission capacity is obtained from the outage probability by so solving $p_o(\lambda) = \epsilon$ for λ. Given that we only have lower and upper bounds on $p_o(\lambda)$ it follows that we will only be able to obtain lower and upper bounds on $\lambda(\epsilon)$. In particular, it is easy to see that the bounds on the transmission capacity are obtained by solving:

$$\begin{aligned} \lambda_l(\epsilon) &= \sup\{\lambda : p_u(\lambda) \le \epsilon\}, \\ \lambda_u(\epsilon) &= \inf\{\lambda : p_l(\lambda) \ge \epsilon\}. \end{aligned}$$

Our bounds on the outage probability are reasonably tight, as shown in the numerical and simulation results section. They are not particularly sophisticated, however, relying only on a simple application of Markov's and Chebchev's inequalities. This simplicity has its advantage, however, in that the bounds

Quantity	Symbol	Value
Transmission radius	$r_{\rm tx}$	10 m
Nominal SINR requirement	β_0	3
Spreading factor	M	16
DS-CDMA SINR requirement	$\beta = \beta_0/M$	$3/16$
Path loss exponent	α	4
Shadowing std. dev.	σ	$\{4, 8\}$

TABLE I

PARAMETER VALUES USED IN NUMERICAL AND SIMULATION RESULTS.

are invertible. This simplicity is critical in obtaining tractable expressions for the transmission capacity. Expressions for transmission capacity under channel variations are presented in [20].

III. NUMERICAL AND SIMULATION RESULTS

We have created a basic ad hoc network outage probability simulator written in Perl to validate the above analysis. Details of the simulation methodology are described in [21]. The parameter values used in obtaining both the numerical and simulation results are shown in Table 1. We assume the transceivers employ DS-CDMA with a spreading factor of $M = 16$. The impact of DS-CDMA on our model is that the nominal SINR requirement, $\beta_0 = 3$ is reduced by a factor of M to $\beta = \beta_0/M = 3/16$. The *normalized SINR requirement*, κ, is defined as $\kappa = \frac{1}{\beta r_{\rm tx}^{\alpha}} = 0.0005\bar{3}$. We varied λ logarithmically from $\lambda_{\min} = 0.00001$ to $\lambda_{\max} = 0.01$.

Figure 1 shows a plot of simulation results of the outage probability $p_o(\lambda)$ versus λ for $i)$ path loss only, $ii)$ path loss with shadowing, $\sigma = 4$ dB, and $iii)$ path loss with shadowing, $\sigma = 8$ dB. All simulation results are 95% confidence intervals with at most 10% relative error. The plot illustrates that, as expected, outages are more likely as more simultaneous transmissions are attempted. Second, we see that the detrimental effect of shadowing on the outage probability: the outage probability is increasing in the shadowing parameter σ.

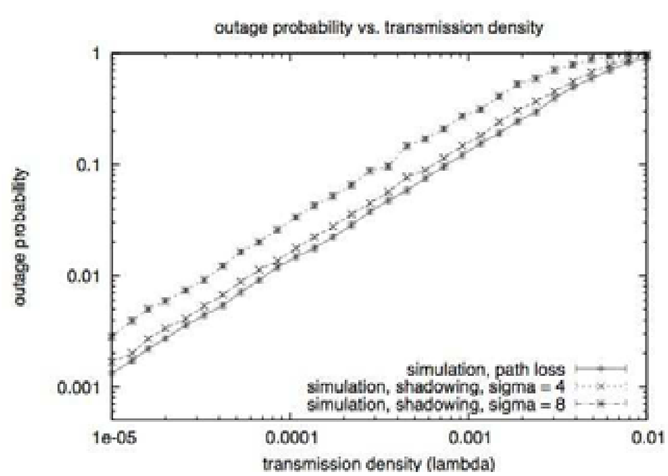

Fig. 1. Simulation results of outage probability $p_o(\lambda)$ versus λ for channels with pure path loss, and channels with path loss and shadowing.

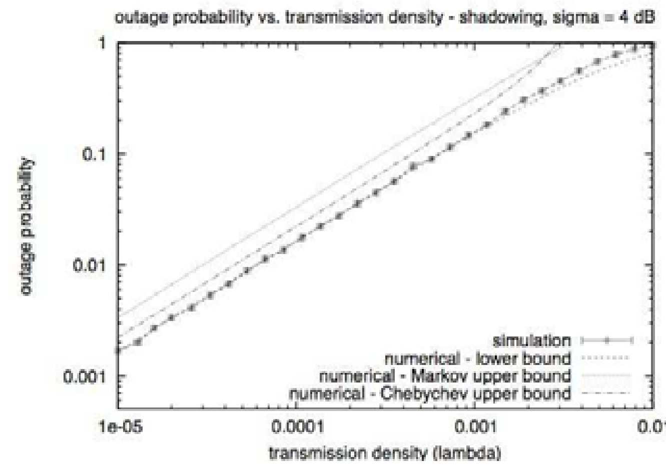

Fig. 2. The lower, Markov upper, and Chebychev upper bounds on the outage probability along with simulation results for shadowing with $\sigma = 4$ dB.

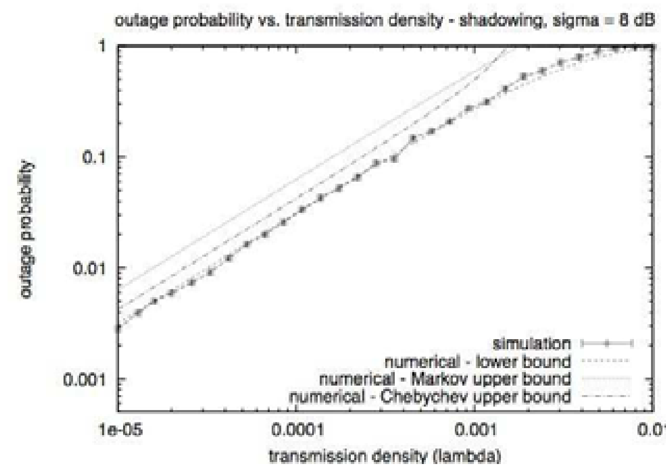

Fig. 3. The lower, Markov upper, and Chebychev upper bounds on the outage probability along with simulation results for shadowing with $\sigma = 8$ dB.

The lower and upper bounds on the outage probability for the shadowing case with $\sigma = 4$ dB, and shadowing with $\sigma = 8$ dB case are shown in Figures 2 and 3 respectively. Several observations may be made from these plots. First, the simulation results lie within the bounds. Second, it is apparent that the lower bound on outage probability is tight. Although difficult to see, the two marks for each simulation result define the confidence interval, and the lower bound lies below or within this interval for almost all of the points in the plots. Recall that the lower bound is given by $\mathbb{P}(\Phi_a^c(\lambda) \neq \emptyset)$ where $\Phi_a^c(\lambda)$ is the set of interfering nodes in $\Phi(\lambda)$ that generate sufficient interference to individually cause an outage for the reference receiver at the origin. The fact that this lower bound is tight means that the principal cause of outage is in fact caused by dominant interferers individually causing outage, as opposed to the cumulative interference from several nodes each of whose interference is small. This observation can be connected to the near-far problem which states that reception is difficult/impossible for nodes whose signal transmitter is far but who are subject to receive interference from a nearby node. Third, we see that the channel variations due to shadowing increase the outage probability, or similarly reduce the transmission capacity. The conclusion is that the diversity provided by channel variations should be exploited, but this requires intelligent scheduling rather than the simple ALOHA-like scheme that is implicit in this paper.

IV. Conclusion

We have presented a stochastic geometric approach to computing the outage probability for a channel model incorporating both path loss and shadowing effects. Numerical and simulation results show that the obtained bounds are reasonably tight. A variety of stochastic geometric "tools" have been employed: marked point processes, void probabilities, Palm measure, and Campbell's Theorem.

There is a great deal of future work that follows naturally from this initial investigation. First, we are in the process of generalizing our bounds to a wide class of channel models, including Rayleigh, Ricean, and Nakagami. Second, we are developing a framework for computing the transmission capacity for this same class of channel models [20]. Third, we are interested in studying distributed control mechanisms where transmitters are assumed to have knowledge of the channel state seen by the receiver (channel side information). More specifically, we are studying how this information can be used to improve the outage probability and transmission capacity, as well as how distributed algorithms to achieve these improvements might be realized in practical network settings.

We are thankful to one of the reviewers for pointing out the connection between our upper bound and the shot noise process. This reviewer also suggested that tighter upper bounds may be obtained from considering higher moments of the shot noise process; we are currently pursuing this approach.

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Appendix

Proof of Theorem 1

Split $\Phi(\lambda)$ into two disjoint complementary processes: $\Phi(\lambda) = \Phi_a(\lambda) \cup \Phi_a^c(\lambda)$, where:

$$\begin{aligned} \Phi_a(\lambda) &= \{(X_i, \Psi_i) \in \Phi(\lambda) : \Psi_i R_i^{-\alpha} > \kappa\}, \\ \Phi_a^c(\lambda) &= \{(X_i, \Psi_i) \in \Phi(\lambda) : \Psi_i R_i^{-\alpha} \leq \kappa\}. \end{aligned}$$

Thus $\Phi_a(\lambda)$ is the set of points that are individually capable of causing outage to the receiver at the origin. Note that although the quantities Ψ_i and X_i are independent for each i in $\Phi(\lambda)$, they are not independent in $\Phi_a(\lambda)$ and $\Phi_a^c(\lambda)$. Also, although $\Phi(\lambda)$ is a *homogeneous* Poisson process of intensity λ, both $\Phi_a(\lambda)$ and $\Phi_a^c(\lambda)$ are heterogeneous (non-stationary) Poisson processes. Define the aggregate interference from these processes as

$$\begin{aligned} Y &= \sum_{i \in \Phi(\lambda)} \Psi_i R_i^{-\alpha}, \\ Y_a &= \sum_{i \in \Phi_a(\lambda)} \Psi_i R_i^{-\alpha}, \\ Y_a^c &= \sum_{i \in \Phi_a^c(\lambda)} \Psi_i R_i^{-\alpha}. \end{aligned}$$

The lower bound on the outage probability is then

$$p_o(\lambda) = \mathbb{P}^0(Y > \kappa) > \mathbb{P}^0(Y_a > \kappa). \tag{4}$$

The upper bound is obtained by noting that the event $\{Y_a \le \kappa\}$ equals the event $\{Y_a = 0\}$ by construction. Conditioning on Y_a relative to κ yields an upper bound:

$$\begin{aligned} p_o(\lambda) &= \mathbb{P}^0(Y_a + Y_a^c > \kappa) \\ &= \mathbb{P}^0(Y_a + Y_a^c > \kappa \mid Y_a > \kappa)\mathbb{P}^0(Y_a > \kappa) + \\ &\quad \mathbb{P}^0(Y_a + Y_a^c > \kappa \mid Y_a \le \kappa)\mathbb{P}^0(Y_a \le \kappa) \\ &\le \mathbb{P}^0(Y_a > \kappa) + \mathbb{P}^0(Y_a^c > \kappa) \end{aligned}$$

The final lower and upper bounds on outage probability are then:

$$\mathbb{P}^0(Y_a > \kappa) \le p_o(\lambda) \le \mathbb{P}^0(Y_a > \kappa) + \mathbb{P}^0(Y_a^c > \kappa). \quad (5)$$

To compute the lower bound we observe that the event $\{Y_a < \kappa\}$ is the same as the event $\{\Phi_a = \emptyset\}$. With this observation we can compute the lower bound using the expression for the void probability of a Poisson process:

$$\mathbb{P}^0(Y_a > \kappa) = 1 - \mathbb{P}^0(Y_a = \emptyset) = 1 - \exp\Big\{-\int_{\mathbb{R}^2} \lambda_a(x)dx\Big\}, \quad (6)$$

where $\lambda_a(x)$ is the density of points in $\Phi_a(\lambda)$ at location x. The density is given by:

$$\lambda_a(x) = \lambda\mathbb{P}(\Psi|x|^{-\alpha} > \kappa) = \lambda\mathbb{P}(\Psi > \kappa|x|^{\alpha}). \quad (7)$$

Noting that the density is radially symmetric, we can switch to polar coordinates, with slight abuse of notation writing $\lambda(r)$ instead of $\lambda(x)$. Also, recall our assumption that we have perfect cancellation in $b(o,1)$. This yields:

$$\mathbb{P}^0(Y_a > \kappa) = 1 - \exp\Big\{-2\pi\lambda\int_1^{\infty} \mathbb{P}(\Psi > \kappa r^{\alpha})rdr\Big\}. \quad (8)$$

Turning now to the upper bound, we compute the moments of Y_a^c via Campbell's Theorem:

$$\begin{aligned} \mathbb{E}^0[Y_a^c] &= \int_{\mathbb{R}^2}\int_{\mathbb{R}_+} \psi|x|^{-\alpha}\lambda_a^c(x,\psi)dxd\psi, \\ \mathrm{Var}^0(Y_a^c) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}_+} \psi^2|x|^{-2\alpha}\lambda_a^c(x,\psi)dxd\psi, \end{aligned}$$

where $\lambda_a^c(x,\psi)$ is the density of Φ_a^c in the point-mark product space $\mathbb{R}^2 \times \mathbb{R}_+$ at point (x,ψ). This density is given by:

$$\begin{aligned} \lambda_a^c(x,\psi) &= \lambda\mathbb{P}(\Psi|x|^{-\alpha} \le \kappa)\frac{d}{d\psi}\mathbb{P}(\Psi \le \psi \mid \Psi|x|^{-\alpha} \le \kappa) \\ &= \lambda f_\Psi(\psi)\mathbb{I}(\psi \le \kappa|x|^{\alpha}). \end{aligned}$$

Using this density in Campbell's Theorem we obtain:

$$\begin{aligned} \mathbb{E}^0[Y_a^c] &= \int_{\mathbb{R}^2}\int_{\mathbb{R}_+} \psi|x|^{-\alpha}\lambda f_\Psi(\psi)\mathbb{I}(\psi \le \kappa|x|^{\alpha})dxd\psi \\ &= 2\pi\lambda\int_1^{\infty} r^{-\alpha}\Big[\int_0^{\kappa r^{\alpha}} \psi f_\Psi(\psi)d\psi\Big]rdr, \\ \mathrm{Var}^0(Y_a^c) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}_+} \psi^2|x|^{-2\alpha}\lambda f_\Psi(\psi)\mathbb{I}(\psi \le \kappa|x|^{\alpha})dxd\psi \\ &= 2\pi\lambda\int_1^{\infty} r^{-2\alpha}\Big[\int_0^{\kappa r^{\alpha}} \psi^2 f_\Psi(\psi)d\psi\Big]rdr. \end{aligned}$$

Using the three constants $p_\Psi, \mu_\Psi, \sigma_\Psi^2$ defined in the statement of the theorem, we can write:

$$\begin{aligned} \mathbb{P}^0(Y_a > \kappa) &= 1 - \exp\{-p_\Psi\lambda\}, \\ \mathbb{E}^0[Y_a^c] &= \mu_\Psi\lambda, \\ \mathrm{Var}^0(Y_a^c) &= \sigma_\Psi^2\lambda. \end{aligned}$$

The Markov and Chebychev inequalities yield upper bounds on $\mathbb{P}^0(Y_a^c > \kappa)$:

$$\begin{aligned} \mathbb{P}^0(Y_a^c > \kappa) &\le \frac{\mathbb{E}^0[Y_a^c]}{\kappa} = \frac{\mu_\Psi}{\kappa}\lambda, \\ \mathbb{P}^0(Y_a^c > \kappa) &\le \frac{\mathrm{Var}^0(Y_a^c)}{(\kappa - \mathbb{E}^0[Y_a^c])^2} = \frac{\sigma_\Psi^2\lambda}{(\kappa - \mu_\Psi\lambda)^2}, \end{aligned}$$

where the Chebychev bound holds provided $\kappa > \mathbb{E}^0[Y_a^c]$. Substituting these bounds yields the upper bounds given in the statement of the theorem.

PROOF OF COROLLARY 1

Let W be zero mean standard normal random variable with variance ν^2. Note that $\Psi = 10^{\frac{W}{10}}$ and hence $W = 10\log_{10}(\Psi)$. Finally, note that

$$Z = W/\nu = \frac{10}{\nu}\log_{10}(\Psi) \sim \mathcal{N}(0,1) \quad (9)$$

is a standard zero mean unit variance random variable. First consider p_Ψ:

$$\begin{aligned} p_\Psi &= 2\pi\int_1^{\infty} \mathbb{P}(\Psi > \kappa r^{\alpha})rdr \\ &= 2\pi\int_1^{\infty} \mathbb{P}\Big(10^{\frac{W}{10}} > \kappa r^{\alpha}\Big)rdr \\ &= 2\pi\int_1^{\infty} \bar{F}_Z\Big(\frac{10}{\nu}\log_{10}(\kappa r^{\alpha})\Big)rdr. \end{aligned}$$

Next consider μ_Ψ. We use the change of variable $z = \frac{10}{\sigma}\log_{10}\psi$ to simplify the inner integral for an arbitrary radius r, denoted $g(r)$:

$$\begin{aligned} g(r) &= \int_0^{\kappa r^{\alpha}} \psi f_\Psi(\psi)d\psi \\ &= \int_0^{\kappa r^{\alpha}} \psi\frac{10/\ln 10}{\sqrt{2\pi}\nu\psi}\exp\Big\{-\frac{(10\log_{10}\psi)^2}{2\nu^2}\Big\}d\psi \\ &= \int_{-\infty}^{\frac{10}{\nu}\log_{10}(\kappa r^{\alpha})} \frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{z^2}{2} + \frac{\ln 10}{10}\nu z\Big\}dz \\ &= \mathbb{E}[\Psi]\int_{-\infty}^{\frac{10}{\nu}\log_{10}(\kappa r^{\alpha})} \frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{(z - \frac{\ln 10}{10}\nu)^2}{2}\Big\}dz \\ &= \mathbb{E}[\Psi]F_Z\Big(\frac{\ln(\kappa r^{\alpha}) - (\frac{\ln 10}{10})^2\sigma^2}{(\frac{\ln 10}{10})\sigma}\Big). \end{aligned}$$

Substituting this into the definition of μ_Ψ gives the expression in the Corollary. Finally consider σ_Ψ^2. Use the same change of variable to simplify the inner integral for an arbitrary radius r, here denoted by $h(r)$, yields

$$h(r) = \int_0^{\kappa r^{\alpha}} \psi^2 f_\Psi(\psi)d\psi = \mathbb{E}[\Psi^2]F_Z\Big(\frac{\ln(\kappa r^{\alpha}) - 2(\frac{\ln 10}{10})^2\nu^2}{(\frac{\ln 10}{10})\nu}\Big). \quad (10)$$